1 Cheeky charges

There is a charged sphere with radius 1/2 (in some units) sitting at the origin. The charge surface density is

$$\sigma(z) = \sigma_0 \left(\frac{1}{2} - z\right)^{-3/2} \tag{1}$$

You can see the total charge is infinite. To obviate, imagine cutting off the infinitely-charged cap $1/2 - \varepsilon < z < 1/2$ and adding a negative point charge at $(0, 0, 1/2)$ with charge opposite the remaining part of the sphere. Then send $\varepsilon \to 0$.

Show this makes sense and find the potential at all points outside the sphere. (Try not to solve any integrals).

2 Relativistic heptagon

Imagine a spaceship capable of relativistic speeds. It accelerates forward with constant proper acceleration a for a certain (unknown) amount of proper time τ. When that time has elapsed, it swiftly rotates by $\pi/2$ left (in its own frame), then again accelerates forward for proper time τ.

It repeats this ritual 7 times, each time accelerating for τ and then rotating left $\pi/2$.

When it's done, it has the same velocity (and orientation) it had originally (with respect to any given inertial frame).

What is the total proper time elapsed?

Hint: replacing 7 above with n, the problem is unsolvable for $n < 4$. For $n = 4$, the total proper time is zero.

3 Jacobi's dream

Put a quantum particle right at the centre of a cubic box of side 1 (in some units). What is the probability (density) to find it back there again after a time t?

Set $m = \hbar = 1$.

4 Space worm

A spaceship is thrusting forward with acceleration a. A naive space worm (modeled as an unelastic string of mass density μ and length l) meets heads on with the spaceship with the intent of getting to know her better. The relative speed when they meet is zero. The ship's solar windshield is unimpressed and the worm gets splattered on it segment by segment. The captain doesn't care and increases thrust as to keep the ship's acceleration a throughout the impact.

What is the profile of the additional thrust as a function of time?

With respect to an inertial frame, how much additional energy has the captain expended because of the worm?

Hint: the equivalence principle.

5 Drag

An object is thrown upwards with initial speed v_0; it's slowed down by a drag force $F = -\lambda \operatorname{sgn}(v) v^2$. Find $v(t)$.

($y(t)$ can also be found, but it's ugly).

6 Spiral

A planet falls onto a much heavier planet following a spiral of the form

$$r \propto \varphi^\alpha \,. \tag{2}$$

Find the number of dimensions of space and the exponent α.